MAT1322
Lecture 9
Sequences

Examples of sequences

(1) 1, 2, 3, 5, 7, 11, 13, 17, ... [prime #s]
(2) 2, 4, 6, 8, 10, ... [$a_n = 2n,\ n \in \mathbb{N}$]
(3) 1, 1, 2, 3, 5, 8, 13, 21, ... [$a_{n+1} = a_n + a_{n-1},\ a_0 = 1, a_1 = 1$]
(4) $3/5,\ -4/25,\ 5/125,\ -6/625,\ 7/3125, \ldots$ [$a_n = (-1)^{n+1} \cdot \frac{n+2}{5^n}$, "alternating sequence"]

What does a sequence do for a large n?
* use a limit, NOTE: a sequence is a function whose domain is $\mathbb{N}$

For (2) and (4):

(2) $\{a_n\} = \{2n\}$

sequence notation

$$\lim_{n\to\infty} 2n = \infty$$

(4) $\{a_n\} = \{(-1)^{n+1} \cdot \frac{n+2}{5^n}\}$

* can't study limit while alternating, so take absolute value.

$$\lim_{n\to\infty} \left| (-1)^{n+1} \cdot \frac{n+2}{5^n} \right|$$

$$= \lim_{n\to\infty} \frac{n+2}{5^n} \qquad \frac{\infty}{\infty} \text{ LH! L'Hopital}$$

$$= \lim_{n\to\infty} \frac{1}{\ln(5) 5^n}$$

$$= \frac{1}{\ln(5)} \cdot \lim_{n\to\infty} \frac{1}{5^n}$$

$$= \frac{1}{\ln(5)} \cdot (0)$$

$$= 0.$$

$$\boxed{\begin{aligned} &\tfrac{d}{dx}(5^n) \\ &= \tfrac{d}{dx}((e^{\ln 5})^n) \\ &= \tfrac{d}{dx}(e^{n \cdot \ln 5}) \\ &= \ln(5) e^n \end{aligned}}$$

PROPERTIES OF SEQUENCES

- a sequence converges if $\lim_{n\to\infty} a_n = A$, $A \in \mathbb{R}$
- if not, it diverges

- a sequence is bounded from above if there exists a number $M \in \mathbb{R}$ such that all $a_n \le M$
- bounded from below if there exists $m \in \mathbb{R}$ such that $m \le$ all a_n
- a sequence is bounded if it's bounded from above and below, otherwise it's unbounded

Series

A series is an infinite sum

In general: $\sum_{n=1}^{\infty} a_n$ We call a_n the coefficient

Ex $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$

partial sums: $3/4$, $7/8$, $15/16$

The partial sums $\sum_{n=1}^{k} \frac{1}{2^n}$ form a sequence.

General formula for $\sum_{n=1}^{k} \frac{1}{2^n} = \frac{2^k}{2^k}$

Now, look what happens when $k \to \infty$:

$$\lim_{k\to\infty} \frac{2^k - 1}{2^k} = \lim_{k\to\infty} \left(\frac{2^k}{2^k} - \frac{1}{2^k} \right) = 1 - \underbrace{\lim_{k\to\infty} \frac{1}{2^k}}_{0} = 1$$

This shows that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

A series is convergent if $\sum_{n=1}^{\infty} a_n = A$, $A \in \mathbb{R}$; otherwise, it diverges

NOTE: If a series diverges, it could mean $\sum_{n=1}^{\infty} a_n = \pm\infty$

OR that the sum doesn't approach any value

Ex $\sum_{n=1}^{\infty} (-1)^n = -1, + 1 - 1 + 1 - 1 + 1 - \ldots$

Partial sums: $-1, 0, -1, 0, -1, 0$

Therefore, it diverges.

Geometric Series

$\sum_{n=1}^{\infty} q^n$ (ex from before: $\sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, so $q = \frac{1}{2}$)

If $q = 1$, series diverges...
If $q = 3$, series diverges...

$\boxed{\text{THEOREM } \sum_{n=1}^{\infty} q^n \text{ converges if and only if } |q| < 1}$

The partial sums $S_k = \sum_{n=1}^{k} q^n$ can be described as: $S_k = \frac{q^{k+1} - 1}{q - 1}$.

To determine if the series converges, look at the limit of the partial sums.

$$\lim_{k \to \infty} \frac{q^{k+1} - 1}{q - 1} \underset{\substack{\uparrow \\ \text{assume } |q|<1}}{=} \lim_{k \to \infty} \frac{0 - 1}{q - 1} = \frac{1}{1 - q}$$

if $q > 1$, limit is ∞
if $q < 1$, limit is 0
if q is (−), limit is 0 if $|q| < 1$ and DNE otherwise.

$$\boxed{\sum_{n=1}^{\infty} q^n = \frac{1}{1-q} \text{ if } |q| < 1}$$